Limits and Their Properties

1.4 Continuity and One-Sided Limits

Objectives

- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Continuity at a Point and on an Open Interval

Continuity at a Point and on an Open Interval

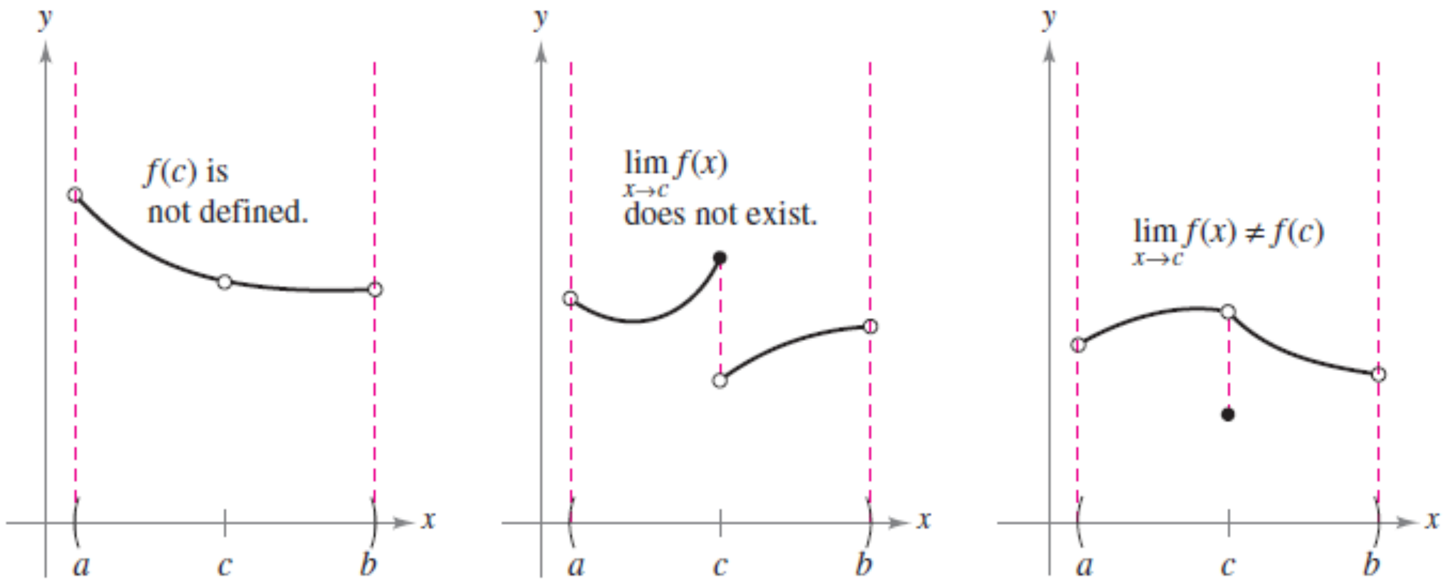
In mathematics, the term *continuous* has much the same meaning as it has in everyday usage.

Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c.

That is, its graph is unbroken at c, and there are no holes, jumps, or gaps.

Continuity at a Point and on an Open Interval

Figure 1.26 identifies three values of *x* at which the graph of *f* is *not* continuous. At all other points in the interval (*a*, *b*), the graph of *f* is uninterrupted and **continuous.**



Three conditions exist for which the graph of f is not continuous at $x = c$.

Figure 1.26

Continuity at a Point and on an Open Interval

In Figure 1.26, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the three conditions is true, the function f is called **continuous at *c*,** as indicated in the following important definition.

Continuity at a Point and on an Open Interval

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

A function is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous.**

Continuity at a Point and on an Open Interval

Consider an open interval *I* that contains a real number *c.*

If a function *f* is defined on *I* (except possibly at *c*), and *f* is not continuous at *c,* then *f* is said to have a **discontinuity** at *c.*

Discontinuities fall into two categories: **removable** and **nonremovable.**

A discontinuity at *c* is called removable when *f* can be made continuous by appropriately defining (or redefining *f*(*c*)).

Continuity at a Point and on an Open Interval

For instance, the functions shown in Figures 1.27(a) and (c) have removable discontinuities at *c* and the function shown in Figure 1.27(b) has a nonremovable discontinuity at *c*.

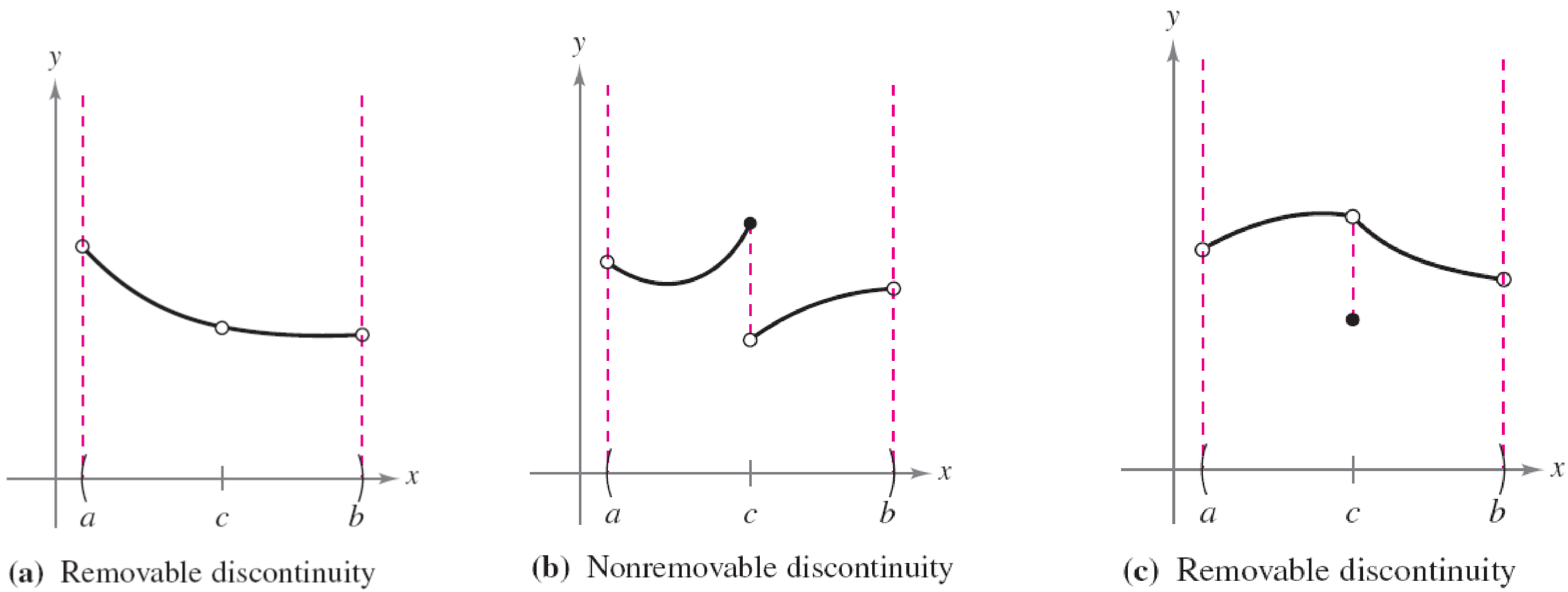


Figure 1.27

Example 1 – *Continuity of a Function*

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$

b. $g(x) = \dfrac{x^2 - 1}{x - 1}$

c. $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$

d. $y = \sin x$

Example 1(a) – *Solution*

The domain of f is all nonzero real numbers.

From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

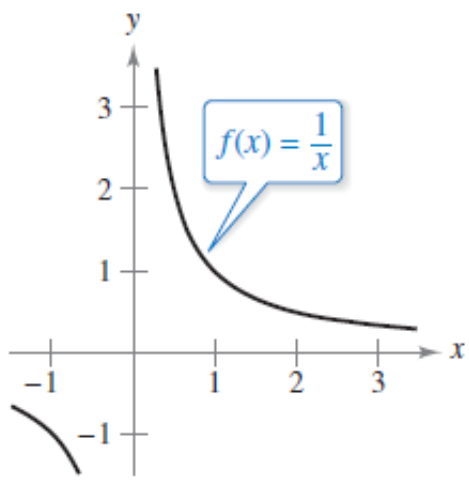
If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 1(a) – *Solution*

cont'd

At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.28(a).



(a) Nonremovable discontinuity at $x = 0$

Figure 1.28(a)

In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

Example 1(b) – *Solution*

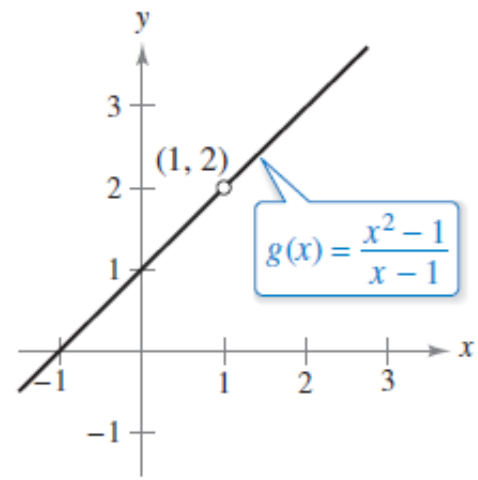
cont'd

The domain of *g* is all real numbers except *x* = 1.

From Theorem 1.3, you can conclude that *g* is continuous at every *x*-value in its domain.

At *x* = 1, the function has a removable discontinuity, as shown in Figure 1.28(b).

By defining *g*(1) as 2, the "redefined" function is continuous for all real numbers.



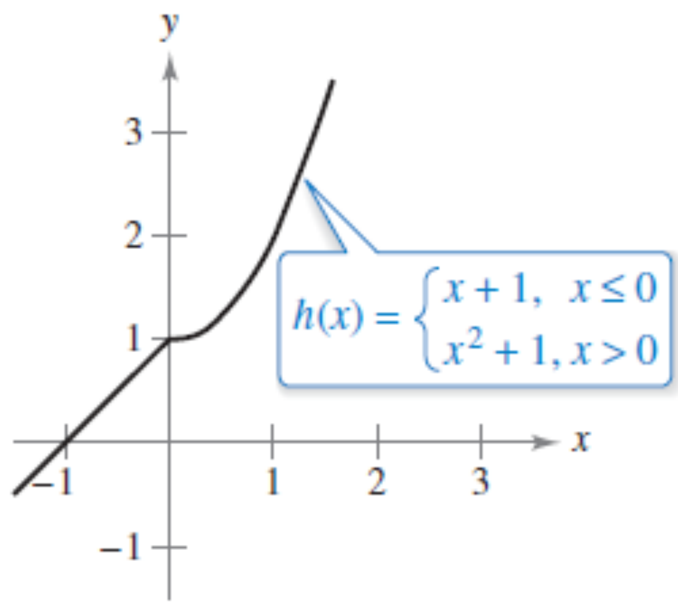
(b) Removable discontinuity at $x = 1$

Figure 1.28(b)

Example 1(c) – *Solution*

cont'd

The domain of *h* is all real numbers. The function *h* is continuous on $(-\infty, 0)$ and $(0, \infty)$, and because $\lim_{x \to 0} h(x) = 1$, *h* is continuous on the entire real number line, as shown in Figure 1.28(c).



(c) Continuous on entire real number line

Figure 1.28(c)

Example 1(d) – *Solution*

cont'd

The domain of *y* is all real numbers.

THEOREM 1.6 Limits of Trigonometric Functions

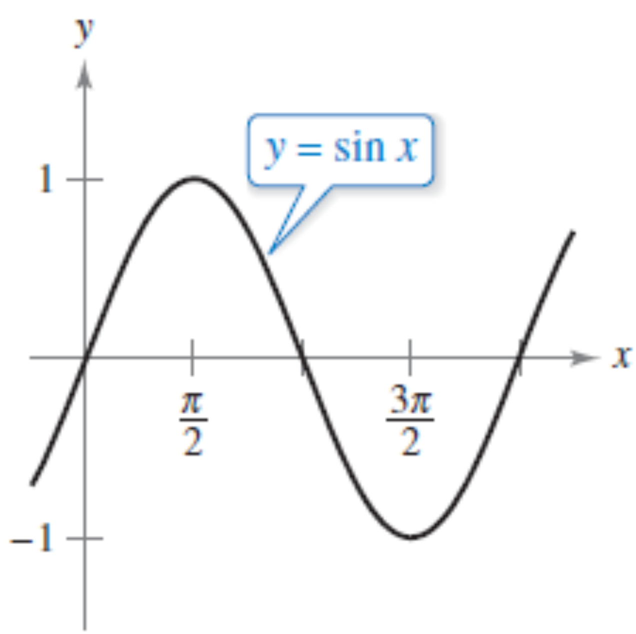
Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \cot x = \cot c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \csc x = \csc c$

Example 1(d) – *Solution*

cont'd

From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.28(d).



(d) Continuous on entire real number line

Figure 1.28(d)

One-Sided Limits and Continuity on a Closed Interval

One-Sided Limits and Continuity on a Closed Interval

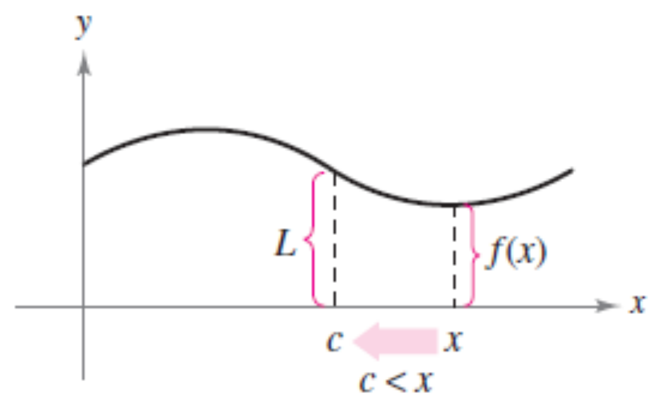
To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.**

For instance, the **limit from the right** (or right-hand limit) means that *x* approaches *c* from values greater than *c* [see Figure 1.29(a)].

This limit is denoted as

$$\lim_{x \to c^+} f(x) = L.$$

Limit from the right



(a) Limit as x approaches c from the right.

Figure 1.29(a)

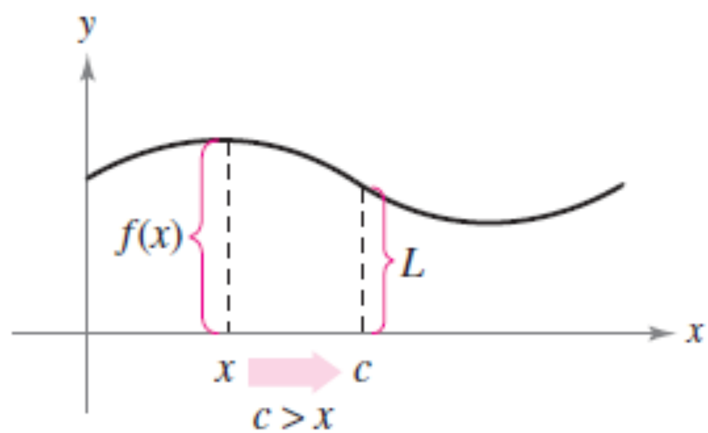
One-Sided Limits and Continuity on a Closed Interval

Similarly, the **limit from the left** (or left-hand limit) means that *x* approaches *c* from values less than *c* [see Figure 1.29(b)].

This limit is denoted as

$$\lim_{x \to c^-} f(x) = L.$$

Limit from the left



(b) Limit as *x* approaches *c* from the left.

Figure 1.29(b)

One-Sided Limits and Continuity on a Closed Interval

One-sided limits are useful in taking limits of functions involving radicals.

For instance, if *n* is an even integer, then

$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

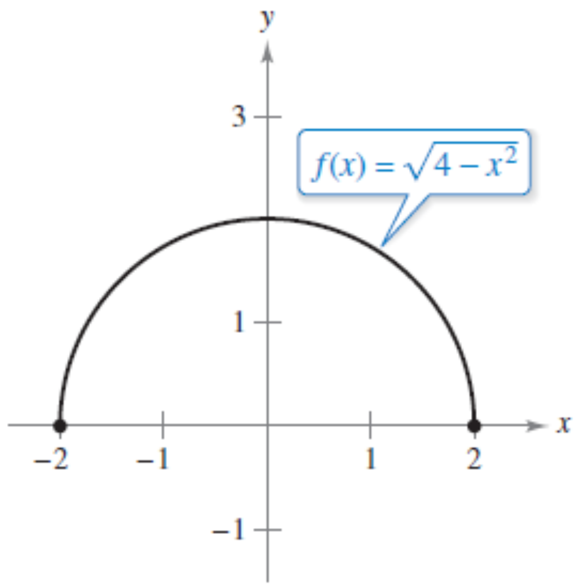
Example 2 – *A One-Sided Limit*

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches –2 from the right.

Solution:

As shown in Figure 1.30, the limit as x approaches –2 from the right is

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.

Figure 1.30

One-Sided Limits and Continuity on a Closed Interval

One-sided limits can be used to investigate the behavior of **step functions.**

One common type of step function is the **greatest integer function** $[\![x]\!]$, defined as

$[\![x]\!] =$ greatest integer n such that $n \leq x$. Greatest integer function

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.

One-Sided Limits and Continuity on a Closed Interval

When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist.* The next theorem makes this more explicit.

THEOREM 1.10 The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

One-Sided Limits and Continuity on a Closed Interval

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals.

Basically, a function is continuous on a closed interval when it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally in the next definition.

One-Sided Limits and Continuity on a Closed Interval

Definition of Continuity on a Closed Interval

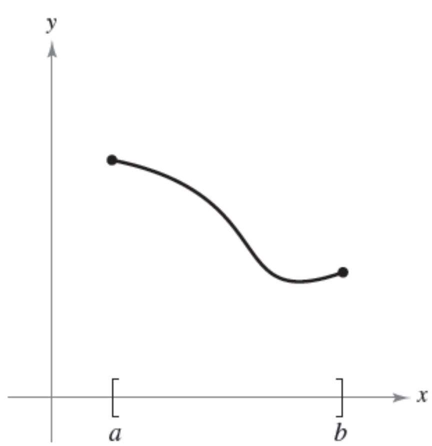
A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.32).



Continuous function on a closed interval

Figure 1.32

Example 4 – *Continuity on a Closed Interval*

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

Solution:

The domain of *f* is the closed interval [–1, 1].

At all points in the open interval (–1, 1), the continuity of *f* follows from Theorems 1.4 and 1.5.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Example 4 – *Solution*

cont'd

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Example 4 – *Solution*

cont'd

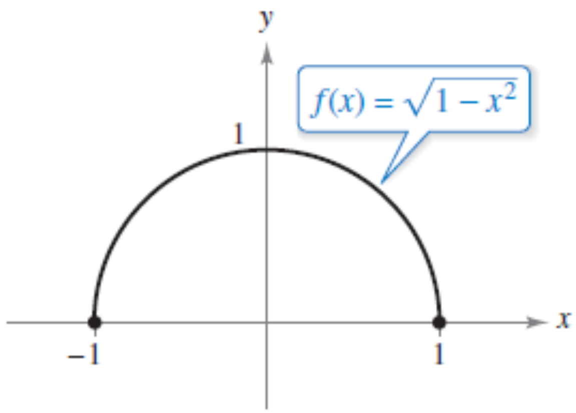
Moreover, because

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that *f* is continuous on the closed interval [–1, 1], as shown in Figure 1.33.



f is continuous on $[-1, 1]$.

Figure 1.33

Properties of Continuity

Properties of Continuity

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}, \quad g(c) \neq 0$

Properties of Continuity

The list below summarizes the functions you have studied so far that are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}, \quad q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this list, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

Example 6 – *Applying Properties of Continuity*

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3 \tan x, \quad f(x) = \frac{x^2 + 1}{\cos x}$$

Properties of Continuity

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \qquad f(x) = \sqrt{x^2 + 1}, \qquad f(x) = \tan \frac{1}{x}.$$

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Example 7 – *Testing for Continuity*

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$

b. $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Example 7(a) – *Solution*

The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

At all other points f is continuous.

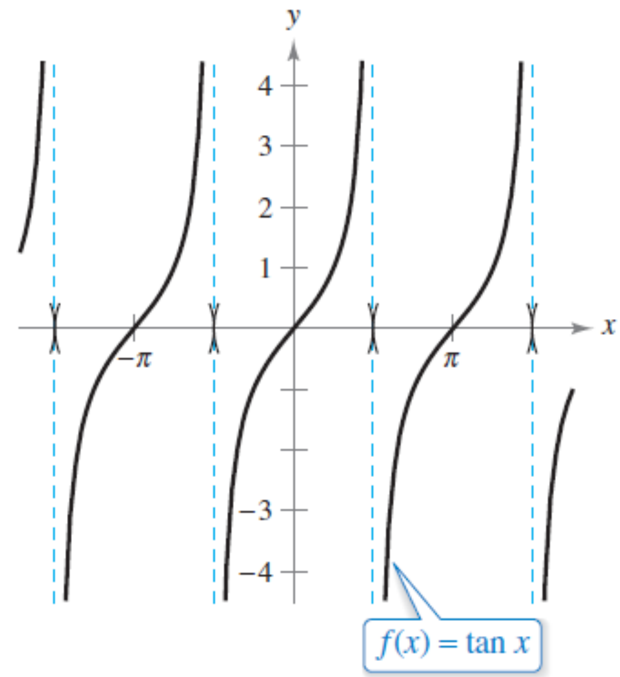
Example 7(a) – *Solution*

cont'd

So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).



(a) f is continuous on each open interval in its domain.

Figure 1.34

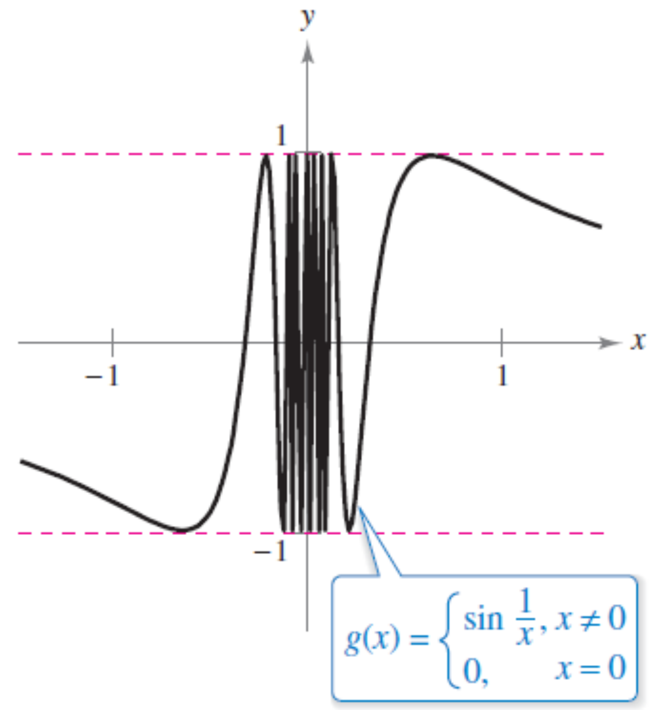
Example 7(b) – *Solution*

cont'd

Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real values except $x = 0$.

At $x = 0$, the limit of $g(x)$ does not exist.

So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).



(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

Figure 1.34

Example 7(c) – *Solution*

cont'd

This function is similar to the function in part (b) except that the oscillations are damped by the factor *x.*

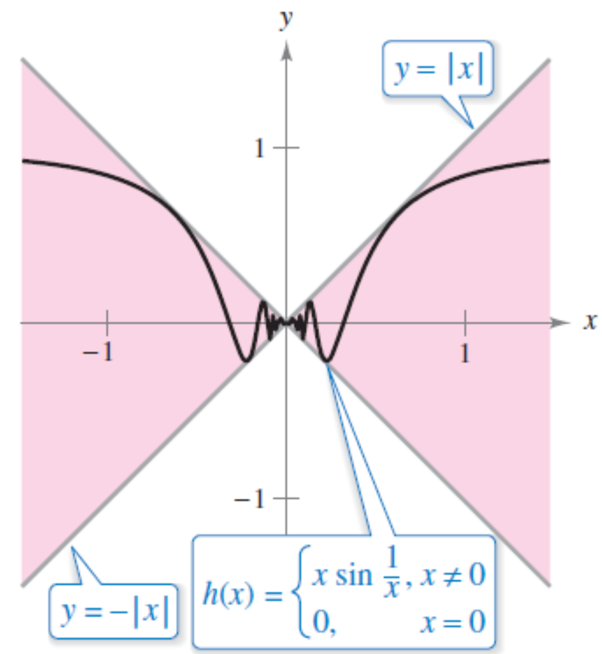
Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin\frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, *h* is continuous on the entire real number line, as shown in Figure 1.34(c).



(c) *h* is continuous on the entire real number line.

Figure 1.34

The Intermediate Value Theorem

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

The Intermediate Value Theorem

The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.**

A proof of this theorem is based on a property of real numbers called *completeness*.

The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, then $f(x)$ must take on all values between $f(a)$ and $f(b)$.

The Intermediate Value Theorem

As an example of the application of the Intermediate Value Theorem, consider a person's height. A girl is 5 feet tall on her thirteenth birthday and 5 feet 2 inches tall on her fourteenth birthday.

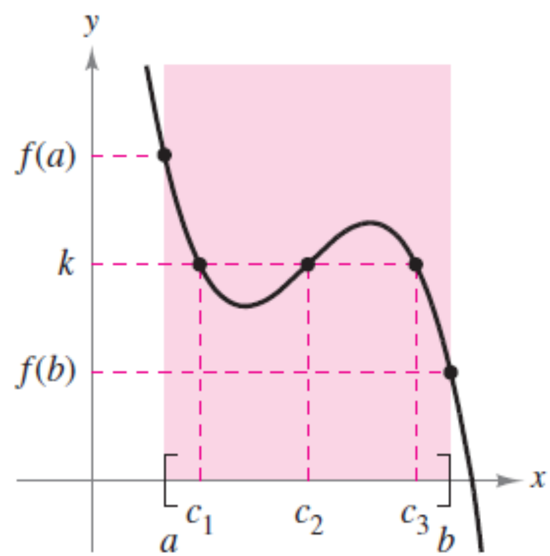
Then, for any height h between 5 feet and 5 feet 2 inches, there must have been a time t when her height was exactly h.

This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

The Intermediate Value Theorem

The Intermediate Value Theorem guarantees the existence of *at least one* number *c* in the closed interval [*a*, *b*] .

There may, of course, be more than one number *c* such that *f*(*c*) = *k* as shown in Figure 1.35.



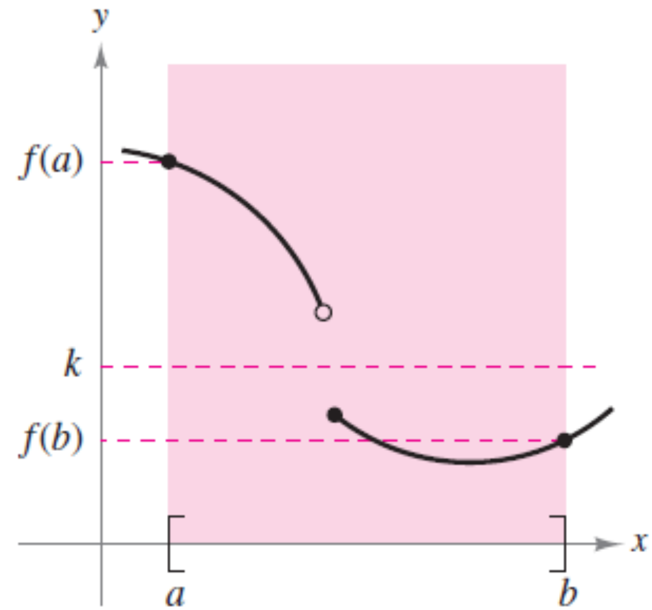
f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]

Figure 1.35

The Intermediate Value Theorem

A function that is not continuous does not necessarily exhibit the intermediate value property.

For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line $y = k$ and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]

Figure 1.36

The Intermediate Value Theorem

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval.

Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

Example 8 – *An Application of the Intermediate Value Theorem*

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1].

Solution:

Note that *f* is continuous on the closed interval [0, 1].

Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$.

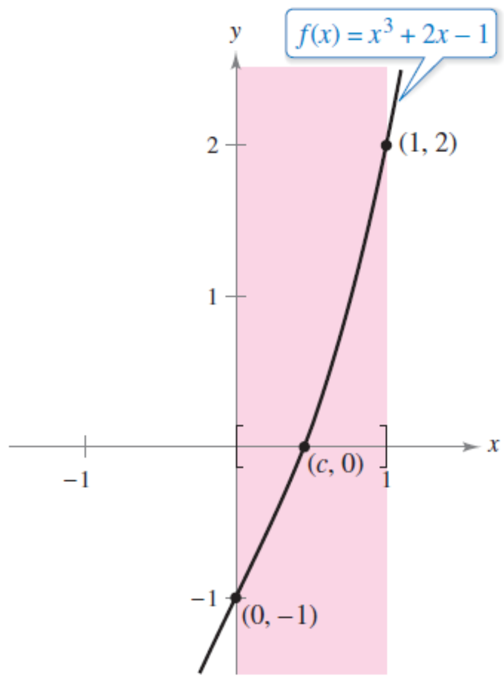
Example 8 – *Solution*

cont'd

You can therefore apply the Intermediate Value Theorem to conclude that there must be some *c* in [0, 1] such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

as shown in Figure 1.37.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.

Figure 1.37

The Intermediate Value Theorem

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8.

If you know that a zero exists in the closed interval [*a*, *b*], then the zero must lie in the interval [*a*, (*a* + *b*)/2] or [(*a* + *b*)/2, *b*].

From the sign of $f([a + b]/2)$, you can determine which interval contains the zero.

By repeatedly bisecting the interval, you can “close in” on the zero of the function.